Commun. Combin., Cryptogr. & Computer Sci., 1 (2021), 160–162

Communications in Combinatorics, Cryptography & Computer Science

Journal Homepage: www.vonneumann-publishing.com/cccs

Break the symmetry in the hierarchical product of an arbitrary graph multiplied by a path or a cycle

Tayyebeh Amouzegar[a,]

[a]Department of Mathematics, Quchan University of Technology,
P.O.Box 94771-67335, Quchan, Iran.

Abstract

In this paper, we investigate the distinguishing number of hierarchical product of an arbitrary graph by a special graph.

Keywords: Distinguishing number, Graph automorphism, Hierarchical product of graphs.

2020 MSC: 05C15, 05C25.

©2021 All rights reserved.

1. Introduction

Albertson and Collins [1] introduced the distinguishing number of a graph. Let G be an undirected simple graph and let r be a positive integer. A coloring $h : V(G) \to \{1, \ldots, r\}$ of the vertices of G is said to be r-*distinguishing* provided no non-trivial automorphism of G preserves all of the vertex color. The *distinguishing number* of G, denoted by $D(G)$, is the smallest integer r such that G has an r-distinguishing coloring. Unless otherwise noted, we apply the notation and phraseology of the book [7] of Bondy and Murty.

In 2009, Barrière, Comellas, Dalfó, and Fiol [5] introduced the hierarchical product of graphs. Several outcomes on the hierarchical product of graphs are obtained, some of which can be seen in [3, 4, 6, 8, 9]. Let G and H be two graphs and H have a root vertex, labeled 0. The *hierarchical product* $G \sqcap H$ is the graph with vertex set $V(G) \times V(H)$ and any two vertices (x_1, y_1) and (x_2, y_2) of $V(G \sqcap H)$ are adjacent if either $x_1 = x_2$ and $y_1y_2 \in E(H)$ or $y_1 = y_2 = 0$ and $x_1x_2 \in E(G)$.

In this paper, the distinguishing number is determined for the hierarchical product of an arbitrary graph by a special graph such as a path graph or a cycle.

2. Main results

The author in [2, Lemma 2.1] has stated automorphisms of the hierarchical product of two graphs. Suppose $\mathfrak{B}(H)$ represents the set of all automorphisms of the graph H that pin the root vertex of H.

Lemma 2.1. *[2, Lemma 2.1] Let G and H be two connected graphs such that $G \neq K_1$. Then*

Email address: t.amoozegar@yahoo.com; t.amouzgar@qiet.ac.ir (Tayyebeh Amouzegar)

Received: November, 1, 2021 Revised: November, 15, 2021 Accepted: November, 20, 2021

$$|\mathrm{Aut}(G \sqcap H)| = |\mathrm{Aut}(G)||\mathfrak{B}(H)|^{|V(G)|}.$$

Theorem 2.2. *Let $G \neq K_1$ be a connected graph with $D(G) \leqslant 2$. Assume that the root vertex of the path P_n is the middle vertex of P_n where n is the odd integer. Then $D(G \sqcap P_n) = 2$.*

Proof. If $D(G) = 1$, then it is easy to see that $D(G \sqcap P_n) = 2$, by using Lemma 2.1. We assume that $D(G) \neq 1$. If we color $G \sqcap P_n$ with less than 2 colors in a distinguishing coloring, then there exists a non-identity automorphism of P_n such as f, such that it preserves the coloring of P_n and f fixes the root vertex of P_n. We can expand f to $G \sqcap P_n$ such that f acts as the identity function on G and obtain a non-identity automorphism of $G \sqcap P_n$ that preserves the coloring of $G \sqcap P_n$, which is a contradiction. Hence, $2 \leqslant D(G \sqcap P_n)$. It remains to show that $2 \geqslant D(G \sqcap P_n)$. First, we color the vertices of G in a distinguishing way with at most 2 colors, because $D(G) \leqslant 2$. Next, we color the vertices in every copy of P_n with 2 colors in a distinguishing way. In view of Lemma 2.1, this coloring is a distinguishing coloring of $G \sqcap P_n$; hence, $2 \geqslant D(G \sqcap P_n)$.

□

Theorem 2.3. *Let $G \neq K_1$ be a connected graph such that $D(G) \gneq 3$.*

(1) *Assume that the root vertex in P_n is the middle vertex of P_n where n is odd. Then $D(G \sqcap P_n) \leqslant x$, where x satisfies the following inequation:*

$$\frac{(x-1)^2(x-2)}{2} \lneq D(G) \leqslant \frac{x^2(x-1)}{2},$$

(2) *Assume that the root vertex in P_n is not the middle vertex of P_n where n is odd. Then $D(G \sqcap P_n) = \lceil \sqrt[n]{D(G)} \rceil$.*

(3) *If n is even, then $D(G \sqcap P_n) = \lceil \sqrt[n]{D(G)} \rceil$.*

Proof. (1) We show that if $\frac{(x-1)^2(x-2)}{2} \lneq D(G) \leqslant \frac{x^2(x-1)}{2}$, then $G \sqcap P_n$ can be colored with at most x colors in a distinguishing way. In view of Theorem 2.2, $2 \leqslant D(G \sqcap P_n)$. If $x = 2$, then $x = 2 \leqslant D(G) \leqslant 4/2 = 2 = x$ and so $x = 2 = D(G)$, which is a contradiction. Thus $x \geqslant 3$. Assume that the vertex set of G will be partitioned to $D(G)$-classes, say, $[1], [2], \dots, [D(G)]$. The vertices of the class $[i]$ are denoted by $v_{i_1}, \dots, v_{i_{s_i}}$ for $i \in \{1, \dots, D(G)\}$. We color the vertices in the class $[i]$ and s_i-copies of P_n to get a distinguishing vertex coloring of $G \sqcap P_n$.

First, we color the vertices of G and P_n as follows:

Step 1. We color all vertices in the class $[i]$, where $1 \leqslant i \leqslant x$, with the color i and the vertices in the s_i copies of P_n with 2 colors in a distinguishing way.

Step 2. We color all vertices in the class $[i]$, where $x+1 \leqslant i \leqslant 2x$, with the color $i - x$ and the vertices in the s_i copies of P_n with 2 colors in a distinguishing way.

Step 3. We color all vertices in the class $[i]$, where $2x+1 \leqslant i \leqslant 3x$, with the color $i - 2x$ and the vertices in the s_i copies of P_n with 2 colors in a distinguishing way.

Continuing these steps, we color all vertices in the class $[i]$, where $(\binom{x}{2} - 1)x + 1 \leqslant i \leqslant \binom{x}{2}x$ with the color $i - (\binom{x}{2} - 1)x$.

Next, suppose that $P_n^{(i)}$ represents the copy of P_n related to the vertex of G that has the color i. Since all vertices in the graph P_n unless the root vertex can be colored distinctly with at least 2 colors in a distinguishing way, so every graph H can be colored by at least $\binom{x}{2}x$ different cases with x colors. Therefore, for all $1 \leqslant i \leqslant x$, there exist at least $\binom{x}{2}x$ graphs $P_n^{(i)}$ in $G \sqcap P_n$ such that those are colored distinctly in a distinguishing way. Hence, the graphs $P_n^{(i)}$, for all $1 \leqslant i \leqslant x$, do not image to each other with some non-trivial automorphism. This way makes a distinguishing coloring for $G \sqcap P_n$ with x colors. Hence, $D(G \sqcap P_n) \leqslant x$.

(2) and (3). By [2, Theorem 3.10]. □

Theorem 2.4. *Let $G \neq K_1$ be a connected graph such that $D(G) \gneq 3$. Then for $n \geqslant 6$, $D(G \sqcap C_n) \leqslant x$, where x satisfies the following inequation:*

$$\frac{(x-1)^2(x-2)}{2} \lneq D(G) \leqslant \frac{x^2(x-1)}{2},$$

Proof. The proof is similar to the Theorem 2.3. □

Theorem 2.5. *Let $G \neq K_1$ be a connected graph with $D(G) \leqslant 2$ and P be the Petersen graph. Then $D(G \sqcap P) = 2$.*

Proof. We color the vertices of G in a distinguishing way with at most 2 colors. Now, we color the vertices in every copy of P with 2 colors in a distinguishing way. In view of Lemma 2.1, this coloring is a distinguishing coloring of $G \sqcap P$; hence, $2 \geqslant D(G \sqcap P)$. Now, we show that $2 \leqslant D(G \sqcap P)$. If we color $G \sqcap P$ with less than 2 colors in a distinguishing coloring, then there exists a non-identity automorphism of P such as f, such that it preserves the coloring of P and f fixes the root vertex of P. We can expand f to $G \sqcap P$ such that f acts as the identity function on G and obtain a non-identity automorphism of $G \sqcap P$ that preserves the coloring of $G \sqcap P$, which is a contradiction. Hence, $2 \leqslant D(G \sqcap P)$. □

Acknowledgment

This research was supported by a grant from Quchan University of Technology. The author would like to thank Quchan University of Technology for the financial support during the preparation of this paper.

References

[1] M. O. Albertson and K. L. Collins, *Symmetry breaking in graphs*, Electron. J. Combin., **3** (1996), no. 1, Research Paper 18, approx. 17 pp. 1

[2] T. Amouzegar, *Distinguishing number of hierarchical products of graphs*, Bull. Sci. math., **168** (2021), 102975, https://doi.org/10.1016/j.bulsci.2021.102975. 2, 2.1, 2

[3] M. Arezoomand and B. Taeri, *Applications of generalized hierarchical product of graphs in computing the szeged index of chemical graphs*, MATCH Commun. Math. Comput. Chem., **64** (2010) 591–602. 1

[4] S. E. Andersona, Y. Guob, A. Tenney and K. A. Wash, *Prime factorization and domination in the hierarchical product of graphs*, Discuss. Math. Graph Theory, **37** (2017) 873–890, doi:10.7151/dmgt.1952. 1

[5] L. Barrière, F. Comellas, C. Dalfó and M. A. Fiol, *The hierarchical product of graphs*, Discrete Appl. Math., **157** (2009), 36–48, doi:10.1016/j.dam.2008.04.018. 1

[6] L. Barrière, C. Dalfó, M. A. Fiol and M. Mitjana, *The generalized hierarchical product of graphs*, Discrete Math., **309** (2009), 3871–3881. doi:10.1016/j.disc.2008.10.028. 1

[7] J. A. Bondy, U. S. R. Murty, *Graph Theory*, Springer, 2008, GTM 244. 1

[8] M. Eliasi, A. Iranmanesh, *Hosoya polynomial of hierarchical product of graphs*, MATCH Commun. Math. Comput. Chem., **69**(1) (2013), 111–119. 1

[9] P. S. Skardal and K. Wash, *Spectral properties of the hierarchical product of graphs*, Phys. Rev. E **94**, 052311 (2016), doi: 10.1103/PhysRevE.94.052311. 1